PERGAMON

International Journal of Heat and Mass Transfer 45 (2002) 2169–2175

International Journal of HEAT and MASS TRANSFER

www.elsevier.com/locate/ijhmt

Slip flow and temperature jump on the impulsively started plate

L.M. de Socio [a,*], L. Marino [b]

[a] Dip. Meccanica e Aeronautica, Via Eudossiana 18, I-00184 Roma, Italy

[b] Dip. Metodi e Modelli Matematici per le Scienze Applicate, Universitá "La Sapienza" Via A. Scarpa 16, I-00161 Roma, Italy

Received 30 March 2001

Abstract

The unsteady compressible boundary layer equations over an impulsively started flat plate are firstly solved subjected to velocity slip and temperature jump conditions. Comparisons are carried out with existing solutions. Then the conjugate problem for a slab of finite thickness is dealt, when the fluid dynamic field is coupled with the thermal field in the solid.

1. Introduction

In this paper the problem of an infinite flat plate which is impulsively started in a compressible gas is firstly revisited. The unsteady laminar boundary layer equations are assumed as the governing physico-mathematical model subjected to slip flow and temperature jump conditions at the wall. To the authors' knowledge this problem was already presented by Patterson in his book [1] on molecular gas dynamics, but there the question was carried out in an oversimplified way in order to obtain an analytical solution. In particular the dissipation function was neglected in the energy equation so that the results lead to unrealistic conclusions although the analysis can be considered as a first approach to a complex situation. The present work was motivated by the increasing attention which is paid to more physically acceptable boundary conditions than the no-slip at the wall when dealing with recent applications of the continuum gas model. Navier [2] in his celebrated paper already presented a velocity slip condition for the motion of a viscous fluid over a body surface. The developments of his idea led to a series of proposals on the same subject which can be mostly found in [3,4], where they are extensively discussed. As it is easily understandable rigorous advances in this field took place following the progress of the molecular kinetics in describing the gas–surface interactions, one model of which can be expressed by the velocity slip and temperature jump relations that will be adopted here [5]. To cite a few cases, more accurate conditions than the usual vanishing of the velocity at the wall are to be imposed in the fluid dynamics of microelectronics and mechanical devices, apart from the possible flight situations of spacecraft, and in all those cases where the continuum Navier–Stokes model must be tempered at the boundaries to take into account the first rarefaction effects. See for example, in this respect, recent papers as those by Beskok [6] and Aoki [7]. The impulsively started plate in a continuum fluid has been extensively studied in the past and the main interest for the problem stays in its simple mathematical representation of a basic physical situation in applied sciences. In this regard the problem may be associated to the Poiseuille and Couette flows for the importance of its results. In particular, when the plate is suddenly put in motion in an incompressible fluid, the problem reduces to an ordinary differential equation after use is made of a similar transformation of the variables, as in the case of the other two cited parallel flows. Analytical solutions for the continuum fluid are thus available in the incompressible case but they can also be found in the compressible one provided that suitable hypotheses are made for the flow and fluid characteristics. When the boundary layer approximation is assumed and a realistic

* Corresponding author.

E-mail addresses: desocio@dma.ing.uniroma1.it (L.M. de Socio), marino@dmmm.uniroma1.it (L. Marino).

0017-9310/02/$ - see front matter
PII: S0017-9310(01)00298-8

Nomenclature

b	slab thickness
l	mean free path
L	reference length
Ma	Mach number
Pr	Prandtl number
Re	Reynolds number
t	dimensionless time for the gas phase
t^*	dimensionless time for the solid phase
T	dimensionless temperature for the gas phase
u	dimensionless velocity component in x direction
U	Stewartson–Dorodnitzyn velocity component in x direction
U_0	wall velocity
v	dimensionless velocity component in y direction
V	Stewartson–Dorodnitzyn velocity component in v direction
x	dimensionless abscissa in the gas phase
y	dimensionless ordinate in the gas phase
y^*	dimensionless ordinate in the solid phase
Y	Stewartson–Dorodnitzyn abscissa

Greek symbols

α	thermal diffusivity of the slab
γ	ratio of specific heats
θ	dimensionless temperature in the slab
λ	thermal conductivity of the gas
μ	absolute viscosity of the gas
Π	coupling parameter
ρ	density

Subscripts

s	quantities evaluated in the solid
t	time derivative
w	quantities evaluated at the wall
y	derivative with respect to y
∞	quantities evaluated at infinity

description of a gas of colliding molecules is taken into account, then a critical approach appears necessary to the validity of the solutions which are obtained in the continuum regime. In a rather old paper, Yang and Lees [8] first addressed their attention to the question of establishing which physico-mathematical model should be considered for the problem of the abruptly started and infinitely thin plate since a macroscopic characteristic length is not present and the only length which is physically meaningful is the microscopic gas mean free path. Those authors began considering the Boltzmann equation and obtained an analytic solution which is valid only for the very first instants after the plate starts, when the flow regime corresponds to that of the free molecules.

To fill somehow the gap between rarefied regimes and continuum flow, Patterson presented a solution for a simplified slip flow model. That solution was obtained via a similarity variable transformation of the boundary layer equations by neglecting – as we said – the dissipation function in the energy equation. It will be shown that Patterson's evaluation of the temperature field leads to an unphysical solution which becomes less and less accurate as the Mach number (or the Reynolds number) increases. On the other hand the solution to the compressible continuum boundary layer equations with no velocity slip and no temperature jump, although taking into account the viscous dissipation, presents the unphysical characteristics of the divergence of both the shear stress and the heat transfer at the wall at the initial time. In the following section the equations which govern the unsteady boundary layer for a compressible gas will be given the form for an incompressible fluid by means of the Stewartson–Dorodnitsyn transformation. A similar variable will then be adopted which changes the energy equation into an ordinary differential equation where the time variable is still present as a parameter. For comparison either the case of no-slip, no-temperature jump and the case of slip and jump but no viscous dissipation will be solved for an isothermal plate and an adiabatic plate which are initially at the same temperature with the ambient. The plate with an adiabatic surface will subsequently deserve more particular comments and will be further discussed in Section 3. In the latter situation the problem will be solved for a slab of finite thickness in order to show effects of the temperature jump which would otherwise be zero for a vanishing thickness.

2. Analysis

The unsteady boundary layer equations for the parallel flow of a compressible gas over a flat plate $y = 0$ are

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial y} = 0, \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y}\right) = \frac{1}{Re}\frac{\partial(\mu \partial u/\partial y)}{\partial y}, \tag{2}$$

$$\rho\left(\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial y}\right) = \frac{1}{(PrRe)}\frac{\partial(\lambda \partial T/\partial y)}{\partial y} + \frac{(\gamma-1)Ma^2}{Re}\mu\left(\frac{\partial u}{\partial y}\right)^2, \tag{3}$$

where the density ρ and the temperature T are made dimensionless with respect to the values at infinity ρ_∞ and T_∞, and the x- and y-components of the velocity u and v, respectively, are made dimensionless with respect to the wall velocity U_0. Moreover the viscosity μ and the heat conductivity λ are dimensionless with respect to μ_∞ and λ_∞. The length L and the time L/U_0 are assumed as references. On the right-hand side of (3) both the thermal conduction and the viscous dissipation are taken into account. Finally γ is the ratio of the specific heats, Re is the Reynolds number, Ma is the Mach number and Pr is the Prandtl number, all calculated in reference conditions. At the initial and at the boundaries

$$u(y,t^-) = v(y,t^-) = 0, \quad [0 \leqslant y \leqslant \infty], \tag{4}$$

$$u(y=0,t^+) = 1 - \xi_1[Kn\partial u/\partial y]_{y=0}, \\ u(y\to\infty,t^+) = 0, \quad v(y=0,t^+) = 0, \tag{5}$$

where we put t^- for $t<0$ and t^+ for $t>0$.

For an isothermal plate at an arbitrary assigned wall temperature T_w. Let

$$T(y,t^-) = 1, \quad [0 \leqslant y \leqslant \infty], \tag{6}$$

$$T(y=0,t^+) = T_w + \xi_2[Kn\partial T/\partial y]_{y=0}, \\ T(y\to\infty,t^+) = 1. \tag{7}$$

In the expressions above Kn is the Knudsen number l/L, where l is the mean free path, and ξ_1, ξ_2 are coefficients which depend on the physical characteristics of the molecules and are usually close to unity [5]. In the following, for the sake of simplicity, we will assume ξ_1 and ξ_2 equal to one, although this assumption is not strictly necessary. Moreover all the solutions presented here correspond to a value of Pr equal to one. This assumption, although not necessary, will be made to carry out the comparison with the existing similar solutions.

As is very often done we apply the Stewartson–Dorodnitzsyn transformation [9]

$$Y = \int_0^y \rho \, dy', \quad U = u, \quad V = \rho v + \frac{\partial Y}{\partial t} \tag{8}$$

to the set of Eqs. (1)–(3) and pertinent conditions (4)–(7) and assume that μ and λ are linear functions of T so that $\mu/(\mu_\infty T) = \sigma = \text{const.}$, with $\sigma = T_w^{1/2}(1+\delta)/(T_w+\delta)$ and $\delta = 1.09924$ [10]. This approach provides an incompressible-like form of the basic equations and in particular one has

$$\frac{\partial U}{\partial t} = \frac{\sigma}{Re}\frac{\partial^2 U}{\partial Y^2}, \tag{9}$$

$$\frac{\partial T}{\partial t} - \frac{\sigma}{RePr}\frac{\partial^2 T}{\partial Y^2} = \frac{(\gamma-1)Ma^2}{Re}\left(\frac{\partial U}{\partial Y}\right)^2. \tag{10}$$

In the following we will take for the reference length $L = l\,|_{y=0}$. Therefore $Kn\,|_{y=0} = 1$ and $Ma = C\,Re$, where C depends upon the gas properties.

The solution to (9) is already well known [1]

$$U = \operatorname{erf}\left[(Y/2)\sqrt{\frac{Re}{\sigma t}}\right] + \exp\left(Y + \frac{\sigma t}{Re}\right) \\ \times \operatorname{erfc}\left[(Y/2)\sqrt{\frac{Re}{\sigma t}} + \sqrt{\frac{\sigma t}{Re}}\right] \tag{11}$$

and from it the dissipation term can be calculated. At this point if one introduces the new variable $\eta = Y/(2t)$ the energy equation becomes

$$\frac{\partial^2 T}{\partial \eta^2} + [4t\eta(RePr)/\sigma]\frac{\partial T}{\partial \eta} \\ = -\left[4Prt^2(\gamma-1)Ma^2/\sigma\right]\left(\frac{\partial U}{\partial Y}\right)^2, \tag{12}$$

where now $T(\eta;t)$ and

$$\left(\frac{\partial U}{\partial Y}\right)^2 = \exp\left[2t\left(2\eta + \frac{\sigma t}{Re}\right)\right]\operatorname{erfc}^2\left[\frac{(t)^{1/2}(\eta + \sigma/Re)}{\sqrt{\sigma/Re}}\right]. \tag{13}$$

The expression (12) above is an ordinary differential equation wherein T appears as a parameter. The conditions on $T(\eta;t)$ are $T(\eta=0,t^+) = T_w + (2t)^{-1}T_\eta\,|_{\eta=0}$, and $T(\eta\to\infty,t^+) = 1$. The solution can then be formally expressed as

$$T(\eta;t) \\ = c_2 + c_1\int \exp\left[-\frac{2RePr}{\sigma}\eta^2 t\right]d\eta - \frac{4(\gamma-1)Ma^2}{Re}t^2 \\ \times \int \exp\left[-\frac{2RePr}{\sigma}\eta^2 t\right]F(\eta;t)\,d\eta,$$

where

$$F(\eta;t) = \int_0^\eta \exp\left[((2RePr)/\sigma)\eta'^2 t\right]U_Y^2(\eta';t)\,d\eta'$$

and c_1 and c_2 are the integration constants. Note that $c_1 = T_\eta(\eta=0,t)$ corresponds to the heat flux at the wall in the transformed variables and $c_2 = T(\eta=0,t)$ to the wall temperature.

All the solutions which will be soon presented here were calculated numerically and the integration was carried out with an accuracy better than 10^{-6} on $T(\eta)$.

Figs. 1 and 2 give the temperature profiles at various times for two values of the Mach number and for $T_w = 1$.

The comparisons which are presented clearly show the noticeable effect of the boundary conditions. Since the velocity slip at the wall corresponds to lower values of the viscous dissipation at a given time with respect to the no-slip condition, this in turn corresponds to lower

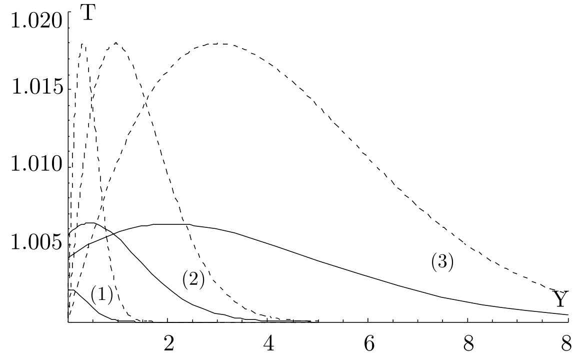

Fig. 1. Temperature distribution vs. Y at different times. $\gamma = 1.4$, $T_w = 1$, $Ma = 0.6$. (1) $t = 0.1$, (2) $t = 1$, (3) $t = 10$. (—) Present solution, (– –) no-slip no-jump.

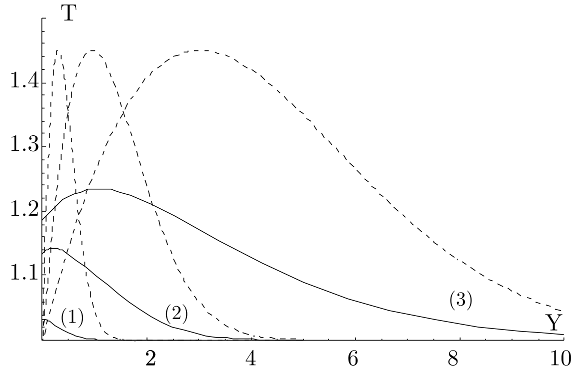

Fig. 2. Temperature distribution vs. Y at different times. $\gamma = 1.4$, $T_w = 1$, $Ma = 3$. (1) $t = 0.1$, (2) $t = 1$, (3) $t = 10$. (—) Present solution, (– –) no-slip no-jump.

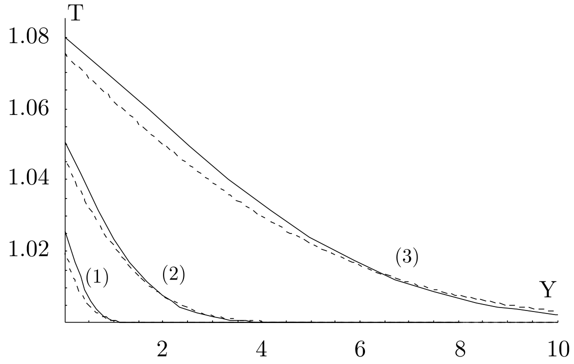

Fig. 3. Temperature distribution vs. Y at different times. $\gamma = 1.4$, $T_w = 1.1$, $Ma = 0.6$. (1) $t = 0.1$, (2) $t = 1$, (3) $t = 10$. (—) Present solution, (– –) [1].

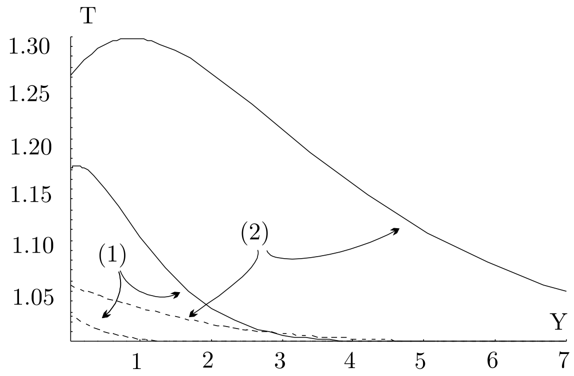

Fig. 4. Temperature distribution vs. Y at different times. $\gamma = 1.4$, $T_w = 1.1$, $Ma = 3$. (1) $t = 0.1$, (2) $t = 1$. (—) Present solution, (– –) [1].

wall temperature values. In addition the temperature jump eliminates the fast initial temperature rise. As $t \to \infty$ the differences at the wall induced by the boundary conditions disappear. Table 1 gives a few comparisons of the heat flux at the wall as calculated according to different boundary conditions.

Figs. 3 and 4 are relative to comparisons between the present results and those calculated after neglecting the viscous dissipation. Note that these comparisons can be carried out only for $T_w \neq 1$, since there is no evolution of the thermal field for $T_w = 1$ following Patterson's assumption. However for low and moderate values of Ma the two sets of solutions are very close at least for small t. Fig. 4 shows instead the great influence of the viscous dissipation at high values of Ma. Table 1 reports the temperature gradient at the wall which is proportional

Table 1
Temperature jump and temperature gradient at the wall vs. time

t	$[T\|_{Y=0} - 1] \times 10^3$	$(\partial T/\partial Y)\|_{Y=0} \times 10^3$	$(\partial T/\partial Y)\|_{Y=0}$
0.001	0.04	0.02	1.28
0.01	0.31	0.17	0.41
0.1	1.76	0.97	0.13
1	4.66	2.51	0.04
10	4.17	2.31	0.01
100	1.91	1.06	0.00

Last column refers to the no-slip, no-temperature jump conditions. $\gamma = 1.4$, $Pr = 1$, $T_w = 1.1$, $Ma = 0.6$.

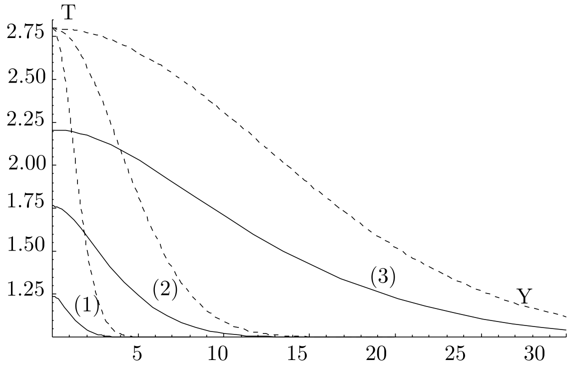

Fig. 5. Temperature distribution vs. Y at different times. $\gamma = 1.4$, $(\partial T/\partial y)|_{w} = 0$, $Ma = 3$. (1) $t = 1$, (2) $t = 10$, (3) $t = 100$. (—) Present solution, (– –) no-slip no-jump.

to the local heat flux. In this respect we recall that the heat flux diverges as $t \to 0$ when there is no velocity slip.

At the conclusion of this section we note that if we assign an adiabatic condition at the wall and neglect the dissipation function then, for an initial wall temperature equal to the temperature of the fluid at rest, the temperature jump at the wall is zero at all times. Therefore no temperature change would ever occur even after the plate starts moving. On the contrary, if the dissipation function is taken into account the resulting temperature distribution in the field can be evaluated following a procedure close to the one for the isothermal plate. In particular, Fig. 5 shows a comparison between the present solution and the one obtained for no-slip and no-jump at the wall. The temperature profile recovers the adiabatic wall temperature at the body surface and tends to the solution for no-jump as $t \to \infty$.

In order to compare our solution with the one for vanishing viscous dissipation in a physical situation, the following section will deal with the case of a slab of a finite thickness which is suddenly started while keeping adiabatic that surface which is not in contact with the gas. The solution for the infinitely thin plate is then a particular case of this more general theory. On the other hand this extension of the Rayleigh problem is coupled with the evaluation of the temperature evolution in the solid. Apart from reducing to the case where the thickness of the slab goes to zero, the solution provides a means for evaluating the thermal field when the surface temperature of the flat plate is a function of the time.

3. The conjugate case

For a slab of finite thickness b which is assumed as a reference length the energy equation is

$$\frac{\partial \theta}{\partial t^*} = \frac{\partial^2 \theta}{\partial y^{*2}}, \tag{14}$$

where the time t^* and the distance y^* have been made dimensionless with respect to b^2/α and b, respectively, and α is the heat diffusivity. The temperature $\theta(y^*, t^*)$ in (14) is dimensionless with respect to T_∞. The gas–solid interface is represented by $y^* = 0$ and y^* points positively into the gas. At $y^* = -1$ we impose either the isothermal condition $\theta(-1, t^*) = \theta_e$ (Case A) or the adiabaticity $(\partial\theta/\partial y^*)(-1, t^*) = 0$ (Case B) whereas the initial condition is always $\theta(y^*, 0) = \text{const}$. A new set of variables has been assumed with respect to (1)–(3) to provide meaningful expressions to the dimensionless quantities in each region of the entire gaseous and solid domain. At the interface a further condition must be satisfied which represents a constraint between the two phases. We recall that the heat flux at $y^* = 0$ must be continuous

$$\theta_e \frac{\partial \theta(0, t^*)}{\partial y^*} = \Pi_1 \frac{\partial T}{\partial y}, \tag{15}$$

whereas (7) and (1) can now be written in the form

$$\theta(0, t^*) = T_w + \left.\frac{\partial T}{\partial y}\right|_{y=0}. \tag{16}$$

In (15) a first coupling parameter appears, namely the characteristic product $\Pi_1 = (\lambda b)/(\lambda_s l)$ where λ_s is the heat conductivity of the solid. At this point the formulation of the coupled differential problem is represented by the set of Eqs. (9)–(14) with the pertinent initial and boundary conditions.

The exact solution to the thermal field in the slab can be found in both Case (A) and Case (B) in books such as [11] as an infinite series, where the unknown temperature at the interface θ_w appears. However, since our interest is mainly in the fluid dynamic aspects of the problem, we recall that in [12] it was shown that excellent approximations to the exact solutions for the solid can be given which greatly simplify the analytic solutions of the fluid dynamic problem without too a cumbersome algebra. In particular, near the interface for small y^*, we write for the temperature distribution in the solid phase the equation

$$\frac{\partial \theta(0, t^*)}{\partial t^*} = 3\left(\frac{\partial \theta(0, t^*)}{\partial y^*} - \theta(0, t^*) - \theta_e\right), \tag{17}$$

when $\theta(-1, t^*) = \theta_e$ and

$$3\frac{\partial \theta(0, t^*)}{\partial t^*} - \frac{\partial^2 \theta(0, t^*)}{\partial \theta^* \partial t^*} = 3\frac{\partial \theta(0, t^*)}{\partial y^*}, \tag{18}$$

when $\partial\theta(-1, t^*)/\partial y^* = 0$. Comparisons with the exact solutions show that these approximate expressions give negligible errors.When (17),(18) are associated to (15), (16), they provide the evolution of the gas temperature at the interface

$$\frac{\partial T(0,t)}{\partial t} = \frac{\partial^2 T(0,t)}{\partial y \partial t} + 3\Pi_2 \left[(\Pi_1 + 1) \frac{\partial T(0,t)}{\partial y} - T(0,t) + \theta_e \right] \quad (19)$$

for an isothermal external wall, and

$$\frac{3}{\Pi_1} \frac{\partial T(0,t)}{\partial t} = (1 + \frac{3}{\Pi_1}) \frac{\partial^2 T(0,t)}{\partial y \partial t} - \Pi_2 \frac{\partial^3 T(0,t)}{\partial y^2 \partial t} + 3\Pi_2 \frac{\partial T(0,t)}{\partial y} \quad (20)$$

for the adiabatic case, where the second coupling parameter Π_2 is the ratio of two characteristics quantities one of the fluid l/U_0 and one of the solid phase b^2/α.

The rate of increase of the wall temperature on the fluid side is directly proportional to the temperature difference $\theta_e - T_w$ through Π_2, and to the heat exchange through the product $\Pi_1 \cdot \Pi_2$.

When Case (A) is considered, the asymptotic steady condition does not depend upon Π_2 but only upon Π_1. This last number is representative of the relative speed of the heat conduction process in the two phases. In Case (B) the asymptotic steady solution does not depend anymore upon both Π_1 and Π_2.

After (19) and (20) are expressed in the η, T variables they represent the coupling conditions for (12) in Case (A) and Case (B), respectively. Figs. 6 and 7 give the comparisons between the temperature distributions as calculated with and without velocity slip and temperature jump at the interface.

Fig. 6 is relative to the case $\theta_e = 1$ and when observed together with Fig. 2 gives a clear picture of the influence of the boundary conditions and of the slab thickness on the thermal field in the gas. We note that in all cases the temperature at the interface is lower at the initial times for the no-slip, no-jump conditions and then increases well above the present solutions. The thickness of the slab, on the other hand, induces higher temperature values than in the case of the thin flat plate. Fig. 7 is relative to $(\partial\theta/\partial y^*)\,|_{y^*=-1} = 0$ and should be observed together with Fig. 5. The difference in the solutions which are induced by the different boundary conditions are smoothed at the initial times by the slab thickness. When the thickness vanishes, the temperature at the interface goes always to the adiabatic wall value as $t \to 0$, when the temperature jump is zero. In all cases $T(Y = 0)$ tends to the adiabatic wall temperature as $t \to \infty$.

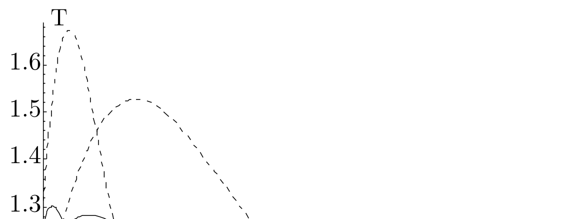

Fig. 6. Temperature distribution vs. Y at different times. $\gamma = 1.4$, $Pr = 1$, $\theta_e = 1$, $Ma = 3$, $\Pi_1 = \Pi_2 = 1$. (1) $t = 10$, (2) $t = 100$. (—) Present solution, (– –) no-slip no-jump.

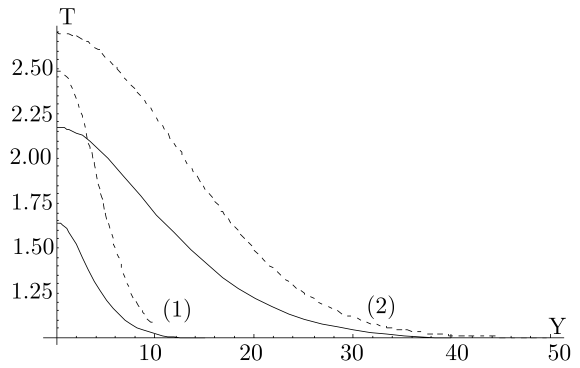

Fig. 7. Temperature distribution vs. Y at different times. $\gamma = 1.4$, $Pr = 1$, $(\partial\theta/\partial y^*)\,|_{y^*=-1} = 0$, $Ma = 3$, $\Pi_1 = \Pi_2 = 1$. (1) $t = 10$, (2) $t = 100$. (—) Present solution, (– –) no-slip no-jump.

4. Conclusions

The Rayleigh problem of the impulsively started flat plate has been revisited as a test situation to show the influence of the velocity slip and temperature jump at the wall. Comparisons are presented of the obtained solutions with other approximate ones which present unphysical singularities. The coupled problem of the thermal fields in the compressible gas and in a slab is solved at the end of the paper to show the influence of the finite thickness.

Acknowledgements

This work was partially supported by the Italian Space Agency through Contract ASI-ARS-99-95, and by the Ministry of the Universities and Scientific and Technological Research.

References

[1] G.N. Patterson, Molecular Flow of Gases, Wiley, New York, 1956.

[2] C.L.M.H. Navier, Memoir sur les lois du mouvement des fluids, CR Acad. Roy. Sci. 17 (1822), 1240 ff.
[3] J Serrin, Mathematical principles of classical mechanics, in: Handbuch der Physik, vol.VIII/1, Springer, Berlin, 1959, 740 ff.
[4] S.A. Shaaf, Chambré, Flow of rarefied gases, in: H.W. Emmons (Ed.), Fundamentals of Gas Dynamics, Princeton University Press, Princeton, NJ, 1958, pp. 718–719.
[5] M.N. Kogan, Rarefied Gas Dynamics, Plenum Press, New York, 1969.
[6] A. Beskok, Physical challenges and simulation of micro fluidic transport, in: 39th AIAA Aerospace Sciences Meeting and Exhibit. AIAA 2001-0719, 2001, Invited lecture.
[7] K. Aoki, Dynamics of rarefied gas flows: asymptotic and numerical analyses of the Boltzmann equation, in: 39th AIAA Aerospace Sciences Meeting and Exhibit. AIAA 2001-0874, 2001, Invited lecture.
[8] H.T. Yang, L. Lees, Rayleigh's problem at low Reynolds number according to the kinetic theory of gases, in: F.M. Devienne (Ed.), Rarefied Gas Dynamics, Pergamon Press, London, 1960, pp. 201–238.
[9] K. Stewartson, in: The Theory of Laminar Boundary Layer in Compressible Fluids, Oxford University Press, London, 1969, 29 ff.
[10] D.R. Chapman, M.W. Rubesin, Temperature and velocity in the compressible laminar boundary layer with arbitrary distribution of surface temperature, J. Aeronaut. Sci. 126 (1949) 547–565.
[11] H.S. Carslaw, J.C. Jaeger, Conduction of Heat in Solids, Oxford University Press, Oxford, London, 1959.
[12] A. Pozzi, E. Bassano, L.M. de Socio, Coupling of conduction and forced convection past an impulsively started infinite flat plate, Int. J. Heat Mass Transfer 36 (1993) 1799–1806.